

The Path-Star Transformation and its Effects on Complex Networks

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(Dated: 3rd Nov 2007)

A good deal of the connectivity of complex networks can be characterized in terms of their constituent paths and hubs. For instance, the Barabási-Albert model is known to incorporate a significative number of hubs and relatively short paths. On the other hand, the Watts-Strogatz model is underlain by a long path and almost complete absence of hubs. The present work investigates how the topology of complex networks changes when a path is transformed into a star (or, for long paths, a hub). Such a transformation keeps the number of nodes and does not increase the number of edges in the network, but has potential for greatly changing the network topology. Several interesting results are reported with respect to Erdos-Rényi, Barabási-Albert and Watts-Strogatz models, including the unexpected finding that the diameter and average shortest path length of the former type of networks are little affected by the path-star transformation. In addition to providing insight about the organization of complex networks, such transformations are also potentially useful for improving specific aspects of the network connectivity, e.g. average shortest path length as required for expedite communication between nodes.

PACS numbers: 89.75.Fb, 02.10.Ox, 89.75.Da

‘There is only one corner of the universe you can be certain of improving, and that is your own self.’ (A. Huxley)

I. INTRODUCTION

The inherent flexibility that complex networks have for representing a wide range of discrete structures and systems has given rise to one of the most powerful and dynamic research subjects [1, 2, 3, 4]. While a vast number of real-world structures have been found to exhibit the small world effect (even the uniformly random Erdos-Rényi – ER – and the scale free Barabási-Albert – BA – complex networks are characterized by small average shortest path), particularly important real-world systems (e.g. protein interaction, WWW and Internet) present scale free organization. Several other types of networks, including those called geographical, have been defined and are subject of current investigations (e.g. [5, 6]).

Each specific category of complex networks exhibit intrinsic properties such as small average degree and high clustering coefficient (small world networks), scale free distribution of node degrees (power law networks), and large average shortest path (geographical networks). The uniformly random ER networks can be well described in terms of their average node degree (most nodes have degree similar to the average). Because of the intrinsic simplicity of the ER networks, this model has served as a reference for characterizing the other models of networks as being relatively *complex*.

Since each type of complex network presents distinctive features, some of them will result particularly efficient for specific tasks. For instance, small world networks are a natural choice for implementing solutions where a relatively short path can be found between any

pairs of nodes, which is an important feature favoring fast communication between pair of nodes. On the other hand, geographical networks tend to present high clustering coefficient, implying a good robustness to edge attacks and large average shortest path length (such networks are among the few structures which are not small world). Several interesting problems arise when one considers the relative efficiency of networks. For instance, one may want to design networks which optimize or suit specific criteria. Another interesting possibility is, given a complex network, to try to modify it in order to improve its fitness with respect to some imposed criteria. One of the few approaches in the complex network literature addressing the latter subject has been described in [7], where the effects of complementing the connectivity by considering several strategies were assessed respectively to improvements in the network resilience.

The modifications performed on a network, e.g. for improving some of its specific features, can be divided into two main categories: (i) *global*, where the modifications are performed indiscriminately throughout the whole network; and (ii) *local*, in which case the modifications are restricted to specific parts of the network. Examples of global modifications are provided by the rewiring schemes which have been traditionally used in complex networks research (e.g. [8]), in which any connection may be changed. An example of local modification is to replace a motif [9] by another, e.g. a triangle by a star (see Figure 1). Observe that the triangle-star motif transformation is directly related to the triangle-star conversion in electrical circuits and, therefore, are particularly interesting for studying flow and random walks in networks [10]. Such motif transformations typically involve the same subset of network nodes.

Network modifications can also be characterized with respect to the respectively implied costs. So, we can

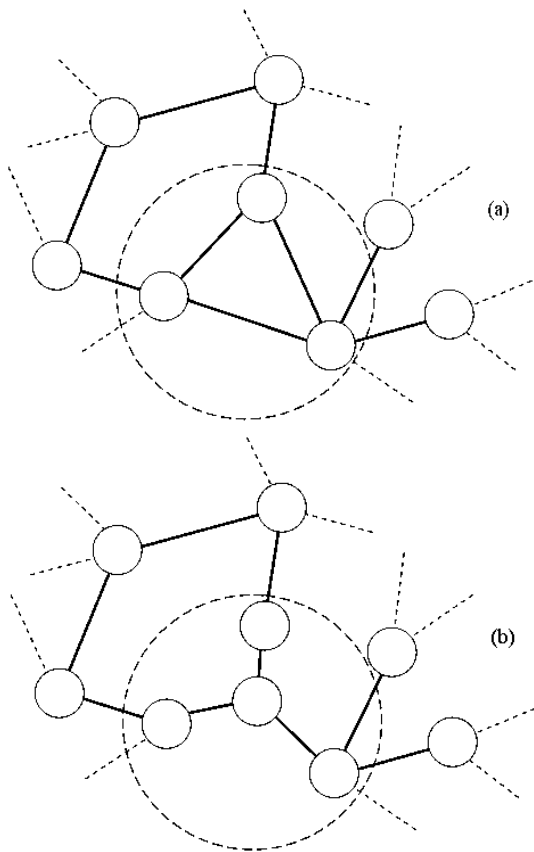


FIG. 1: Example of motif transformation: the triangle-star conversion in a complex network: the triangle structure, shown within the circle in (a), is transformed into the encircled star in (b). Note that such a conversion adds a new edge to the network.

define the following two main categories: (a) *costless*, involving no (or almost no) cost; and (b) *expensive*, in which case the modifications have a relevant cost. The cost is specific to each situation. For instance, the addition of nodes in the Internet will involve costs related to the acquisition of new computers (servers). Even modifications preserving the number of edges and nodes (e.g. rewirings) may involve costs related to the physical change of the connections, the period of time the system will be down, as well as administrative expenses.

A critical research issue regards the effect of the modifications on the network topological and dynamical characteristics. For instance, the rewiring scheme in [8] changes several network features, but does not affect the degree distribution. Therefore, it is interesting to characterize as completely as possible the effects of network modifications. This can be done by considering several measurements [4], especially those which are more closely related to the expected features. Interestingly, as corroborated by the results reported in this work, the same type of modification can have substantially different effects of

the topological features of the a network depending on the category of the latter.

The present work investigates the effect of a special type of network modification, especially with respect to improvements of specific features of the network. The considered modification is local and involves replacing paths [15] by stars, as illustrated in Figure 2. Such a type of motif transformation can be understood as being related to the generalization of the path star conversion in electrical circuits. More specifically: a path is identified in the network; its middle node is determined; and the connections among the nodes in the path are replaced so that the middle node becomes directly connected to all remainder nodes in the original path. Observe that such a path-star motif transformation [16] preserves the number of nodes and does not increase the number of edges (note that eventual edges already existing between the middle point and any of the other nodes will become redundant), therefore implying low or no cost. The inverse motif transformation, namely taking a star into a path may also be considered for improvement of complex networks.

The choice of the type of local transformation considered in this work, among a large variety of alternative possible modifications deserves some justification. First, it implies a major alteration of the topology of the network while preserving the number of nodes and not increasing the number of edges. Second, the two involved motifs – i.e. paths and stars – are among the most important substructures underlying the connectivity of networks. Paths are also intrinsically related to random walks on networks, while stars are related to hubs. In addition, they are also very different one another and can even be considered dual. Paths have an intrinsic sequential, linear nature. Contrariwise, stars are centralized motifs. Also, the path-star (as well as its inverse) preserves the overall connectivity of the overall network, in the sense of not producing additional connected components. The effects of the path-star transformation (e.g. for reducing the average distance between pair of nodes) depends on the connectivity of the nodes involved in the chosen path and on the type of network. In the simplest case where the original network involves only the path, the effects of the path-star transformation is clear and include: (i) the average shortest path length (as well as the network diameter) is reduced; (ii) the node degree distribution changes, giving rise to a hub; and (iii) the clustering coefficient remains zero. Interestingly, the middle node remains with the higher betweenness centrality. Figure 2 illustrates a more generic path-star transformation. Observe that, unlike the triangle-star conversion, this transformation does not increase the number of edges because the middle point is used as center of the star. Also, in case of long paths, the path-star transformation will give rise to a hub. Such a duality between long paths and hubs can even be related to the justification of the presence of the latter in some types of networks (i.e. hubs would be the result of the path-star

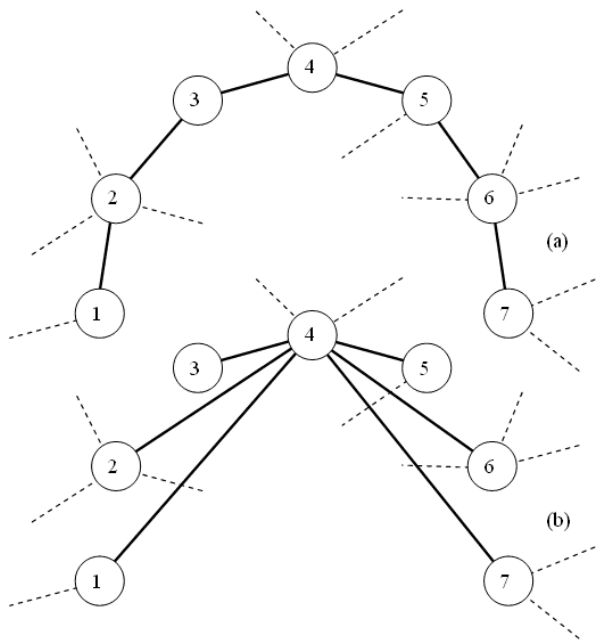


FIG. 2: The transformation of a path (a) into a star motif (b) can be obtained by replacing the edges along the path so that they connect the middle with the remainder nodes. Observe that such a transformation keeps the number nodes and does not increase the number of edges, but has an important effect in decreasing most of the distances between the nodes. In the case where edges originally interconnect the nodes in the original path, the path-star transformation may also tend to increase the clustering coefficient.

transformation required for optimization of specific network features).

Although the effects of the path-star motif transformation can vary with the specific chosen path and type of network, it is expected that – when applied to networks involving relatively long paths, this transformation will tend to reduce the average shortest path length and give rise to hubs. The characterization of the effect of this transformation provides one of the main motivations for the present article. The inverse motif transformation, namely star-path is expected to have opposite effects.

This article starts by presenting the basic concepts and methods and follows by presenting and discussing the results from the perspectives of the average and standard deviation of the diameter, clustering coefficient and average shortest path length as well as correlations between such measurements and the length of the chosen path.

II. BASIC CONCEPTS AND METHODOLOGY

This work focuses undirected networks, which can be represented by symmetric adjacency matrices K , such that each existing edge (i, j) implies $K(i, j) = K(j, i) =$

1, with $K(i, j) = K(j, i) = 0$ indicating absence of the respective edge. Each network contains N nodes and E edges. The *degree* of a node i , expressed as k_i , is equal to the number of edges attached to that node. The average node degree considering the whole network is henceforth expressed as $\langle k \rangle$. The *immediate neighbors* of a node i are those nodes which are directly connected to i . The *clustering coefficient* of a node i , henceforth represented as cc_i , is given by the ratio between the total number of existing edges between the immediate neighbors of i and the maximum number of possible edges between those nodes. The average clustering coefficient considering the whole network is henceforth expressed as $\langle cc \rangle$. Two edges are *adjacent* whenever they share one of their extremities. A sequence of adjacent edges form a *walk*, whose length corresponds to the number of involved edges. A *path* is a walk where the edges and nodes never repeat themselves. The *shortest path* between two nodes correspond to one (or more) of the paths between those two nodes which involve the smallest possible number of edges. The average shortest path length considering the whole network is henceforth expressed as $\langle spl \rangle$. The *diameter* of the network, abbreviated here as *diam*, corresponds to the length of the longest shortest path between any pair of nodes in the network.

This work considers the effect of the path-star transformation in three principal types of networks: ER, BA and WS. The BA networks were grown by starting with m_0 nodes and incorporating new nodes with m edges and preferential attachment (e.g. [1]), so that the average node degree is given as $\langle k \rangle = 2m$. The ER networks were obtained by implementing connections between each pair of distinct nodes with fixed probability. The WS networks were obtained by starting with a cycle of N nodes and rewiring 20% of the edges. The parameters of the models were chosen so as to yield similar node degrees for all three models (e.g. [11, 12]).

The experiments reported in this work involved a single path-star transformation performed on a randomly chosen path. More specifically, a node i of the network is randomly selected and a random walk performed until no more unvisited nodes are available. Another similar random walk is then performed starting again from i , in order to explore the other extremity of the path. The so-obtained path is then transformed into a star by removing the edges along the path and placing then between the middle node and all remainder nodes. Observe that the statistical relevance of the adopted experimental procedure increases with the length of the selected paths.

The effects of the path-star transformation in changing the network topology can be quantified in terms of ratios between measurements taken after and before the transformation. The present work considers the following ratios:

$$R(diam) = diam(after)/diam(before) \quad (1)$$

$$R(cc) = \langle cc \rangle (after) / \langle cc \rangle (before) \quad (2)$$

$$R(spl) = \langle spl \rangle (after) / \langle spl \rangle (before) \quad (3)$$

III. RESULTS AND DISCUSSION

A total of 50 simulations were performed for each network of type ER, BA and WS models with respect to each of the four following configurations: (a) $N = 100$ and $m = 3$; (b) $N = 100$ and $m = 5$; (c) $N = 200$ and $m = 3$ and (d) $N = 200$ and $m = 5$. The obtained results are presented and discussed in the two following subsections with respect to their average/standard deviation and correlations with the initial path length.

A. Average and Standard Deviation of Ratios

The average and standard deviation of the ratios obtained for the changes in diameter, clustering coefficient, average shortest path length and length of the initial chosen path are given in Table I with respect to each of the above four configurations and the three types of considered network models.

Table I also shows, in its last group, the values of the lengths of the randomly chosen paths used for the path-star transformation. Such lengths tended to present a relatively small dispersion, especially for the cases involving $m = 5$, while the smallest dispersions were obtained for the WS networks. Such relatively small variations are interesting by themselves as could be expected that the random choice of the initial node for the walk would imply paths with markedly different lengths. However, the relatively long obtained path lengths, especially for the ER and BA cases, implied higher probability of taking paths involving the same nodes. The average path lengths obtained for the ER networks tended to be about twice as large as the respective path lengths obtained for the BA model. The path lengths resulting for the WS networks were almost identical to the number of respective nodes in those structures. The resulting small dispersion of path lengths enhances the statistical significance of the analysis of measurement changes described in the following.

First, we consider the ratios of changes in the diameter of the network, $R(diam)$ (see first group in Table I). Small standard deviations were again obtained for all such measurements. In the average, the path-star transformation had relatively little effect in changing the average diameter in the ER networks and BA, but implied major reduction of diameter in the case of the WS model, an effect which tended to increase for larger m . The substantial reductions of diameter obtained for the WS networks are a consequence of the long initial paths char-

acterizing this case. Interestingly, the path-star transformation almost unaffected both the ER and BA networks.

Regarding the ratio of changes in the clustering coefficient ($R(cc)$), shown in the second group in Table I), relatively small dispersions were again obtained, except for the ER networks with $m = 3$. In addition, almost no alteration was again obtained for the average ratios in the case of the BA networks. Decreases of about 20% in the clustering coefficient were obtained for the WS networks. The changes did not significantly vary with N or m . Relatively high increases (all larger than 2) of clustering coefficient were implied by the path-star transformation in the case of the ER model. This is a consequence of the fact that the concentration of the edges along the path into the middle node tend to complement the previous connections between non-subsequent nodes in the path, giving rise to triangles and consequently higher clustering coefficient. This effect is less intense for the BA networks because less edges are typically found between non-subsequent path nodes in that case (at least for the considered average degrees).

The ratio of changes in the average shortest path lengths are given in the third group of results in Table I). Small dispersions of these ratios were yet again generally observed, enhancing the statistical significance of the analysis. While almost no changes in average path lengths were observed for the ER and BA models, substantial decreases were implied by the path-star transformation in the case of the WS networks. Though the latter result could be expected (the WS involves an initial long path and large diameter), the relatively small decrease of average shortest path length obtained for the ER model was somewhat surprising. As with the clustering coefficient, the effect of the path-star transformation tended to become more accentuated for larger m in the case of the WS networks.

All in all, the above results contained a series of expected and surprising facts. Remarkably, relatively small dispersions were obtained for the initial path length, enhancing the statistical relevance of our simulations. Because the BA model involves relatively short initial paths and is characterized by having several links connected to hubs, it would be only too natural to expect that the path-star transformation would have relatively little effect on that model. On the other hand, because of the initial sequential nature of the WS structures, it would be reasonable to expect that the path-star would imply substantial changes in diameter, clustering coefficient and average shortest path length for that model. The results obtained for the ER contained surprises, especially regarding the large change of clustering coefficient and small changes of diameter and average shortest path length observed for those networks. The small changes in diameter and average shortest path length obtained for the ER networks are particularly unexpected because the initial path lengths tended to be twice as larger in the ER as compared to the BA structures.

| Measurement | Configuration | ER | BA | WS |
|-------------|------------------|--------------------|-------------------|--------------------|
| $R(diam)$ | $N = 100, m = 3$ | 0.932 ± 0.005 | 0.996 ± 0.100 | 0.26 ± 0.060 |
| | $N = 100, m = 5$ | 0.880 ± 0.126 | 1.003 ± 0.084 | 0.399 ± 0.009 |
| | $N = 200, m = 3$ | 0.940 ± 0.085 | 1.00 ± 0.00 | 0.254 ± 0.161 |
| | $N = 200, m = 5$ | 0.964 ± 0.078 | 1.00 ± 0.00 | 0.34 ± 0.020 |
| $R(cc)$ | $N = 100, m = 3$ | 2.479 ± 1.136 | 1.100 ± 0.189 | 0.858 ± 0.025 |
| | $N = 100, m = 5$ | 2.375 ± 0.353 | 1.003 ± 0.084 | 0.794 ± 0.010 |
| | $N = 200, m = 3$ | 2.893 ± 1.792 | 1.140 ± 0.179 | 0.861 ± 0.032 |
| | $N = 200, m = 5$ | 3.396 ± 0.599 | 1.300 ± 0.179 | 0.790 ± 0.006 |
| $R(spl)$ | $N = 100, m = 3$ | 0.916 ± 0.045 | 0.996 ± 0.010 | 0.480 ± 0.022 |
| | $N = 100, m = 5$ | 0.909 ± 0.028 | 0.990 ± 0.012 | 0.686 ± 0.012 |
| | $N = 200, m = 3$ | 0.912 ± 0.056 | 0.996 ± 0.011 | 0.416 ± 0.089 |
| | $N = 200, m = 5$ | 0.875 ± 0.025 | 0.984 ± 0.013 | 0.592 ± 0.006 |
| L | $N = 100, m = 3$ | 49.72 ± 15.20 | 20.78 ± 9.31 | 99.58 ± 2.55 |
| | $N = 100, m = 5$ | 81.72 ± 11.63 | 45.60 ± 11.27 | 100.00 ± 0.00 |
| | $N = 200, m = 3$ | 73.18 ± 32.56 | 34.48 ± 13.76 | 193.48 ± 25.96 |
| | $N = 200, m = 5$ | 147.76 ± 14.83 | 79.64 ± 18.84 | 200.00 ± 0.00 |

TABLE I: The average \pm standard deviation obtained for the ratios of change of diameter ($R(diam)$), clustering coefficient ($R(cc)$), and average shortest path length ($R(spl)$) with respect to each of the types and configurations of networks. See text for discussion.

B. Correlations between Ratios and Initial Path Lengths

Despite the relatively small standard deviation of the initial path lengths obtained for the considered models, it is interesting to investigate the relationships between those values and the three considered ratios $R(diam)$, $R(cc)$, $R(spl)$. Because the diameters involve only a few integer values (e.g. 2, 3, etc.), such correlations are not considered here. The scatterplots obtained by plotting $R(cc)$ and (spl) against L are presented in Figures 3 and 4, respectively. Each of these two figures includes all the 50 realizations obtained for each of the configurations of the three considered network models.

As can be appreciated from Figure 3, the ratio of changes in the clustering coefficient, as implied by the path-star transformation, tended to increase substantially with L in the case of the ER and BA models, while very similar (and small) ratios were obtained for the WS networks. Interestingly, the increase for ER and BA is more accentuated for $m = 3$ than for $m = 5$. Observe also that the dispersion of the ratios $R(cc)$ obtained for the ER and BA models tended to increase substantially with L . This effect is particularly counter-intuitive in the sense that the path-star transformations performed on particularly long initial paths (especially in the ER case) — which therefore encompass most of the network nodes — could be expected to lead to paths containing similar nodes and, therefore, similar connectivity.

Figure 4 shows the scatterplot obtained by plotting

the ratios of change in the average shortest path length, i.e. $R(spl)$, in terms of L . The path-star transformation tends to imply a reduction of the average shortest path (i.e. $R(spl) < 1$) in all cases, which more intense effects for larger values of the initial path length L . Unlike what was found for the clustering coefficient, relatively small dispersions of $R(spl)$ are observed for each value of L . The behavior of $R(spl)$ is completely different from those obtained for the ER and BA cases. More specifically, the reductions of $R(spl)$ observed for the WS networks are much more accentuated and evolve in a distinct fashion when L increases. Actually, an almost linear decrease of $R(spl)$ can be identified in Figure 4(c).

IV. CONCLUDING REMARKS

Two of the most important connectivity patterns characterizing complex networks are paths and hubs, which have a dual nature. The present work has investigated how the transformation of paths into stars (hubs in the case of long paths) in networks tend to change the properties of ER, BA and WS network as far as the properties of diameter, clustering coefficient and average shortest path length are concerned. For each considered network, a path was randomly chosen and transformed into a star. Ratios between the above measurements obtained after and before the transformation were calculated in order to allow the identification of the major effects implied by the path-star transformation with respect to each of the three

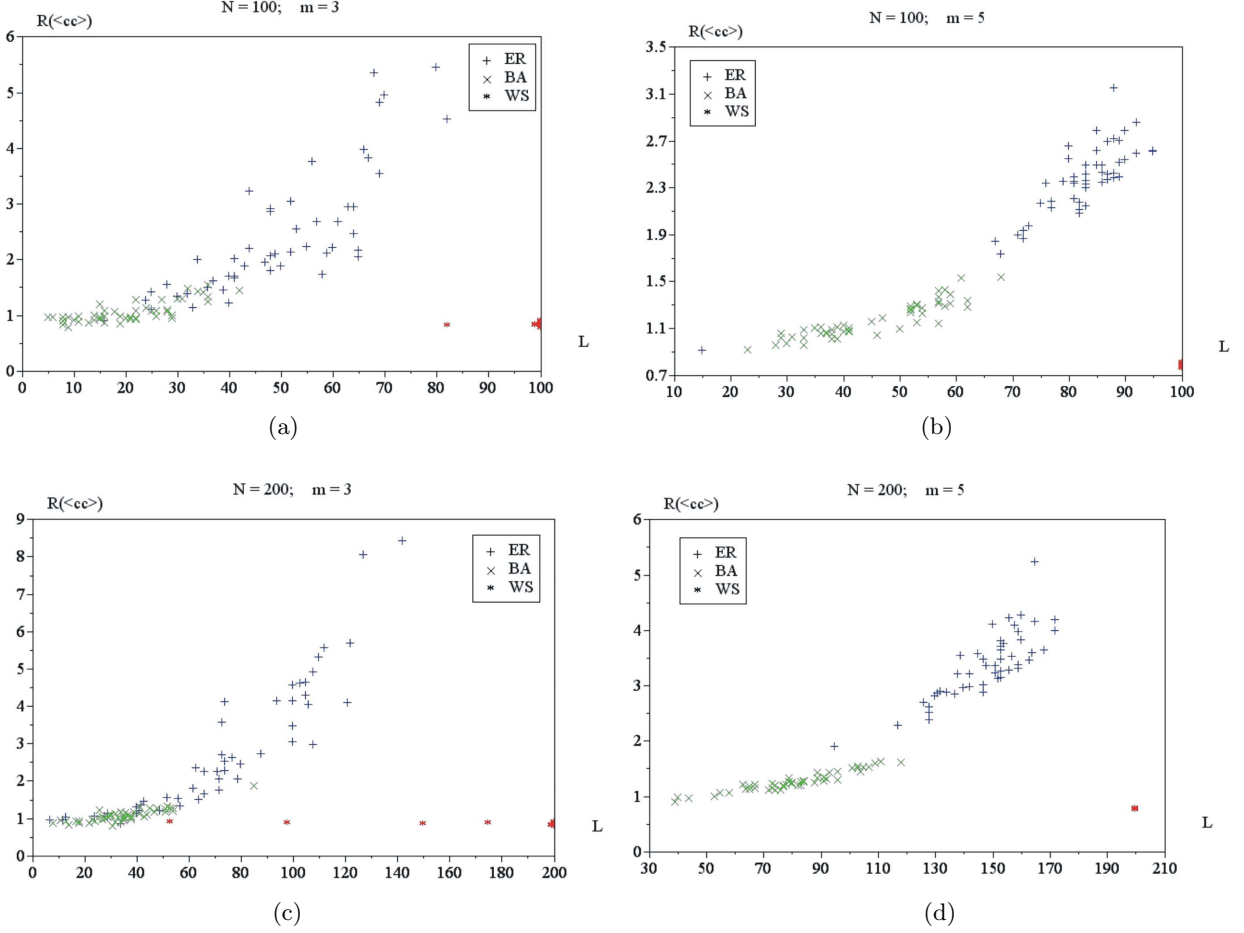


FIG. 3: The scatterplots obtained by plotting the ratio of changes in the clustering coefficient, $R(cc)$, in terms of L for the four considered configurations, i.e.: $N = 100, m = 3$ (a); $N = 100, m = 5$ (b); $N = 200, m = 3$ (c) and $N = 200, m = 5$ (d). This scatterplot indicates a well-defined tendency of $R(cc)$ to increase with L in the case of the ER and BA networks, while almost identical ratios were obtained for the WS model.

types of networks. A series of interesting results were obtained which provide insights not only about the effects of the transformation but also on the inherent structure of the connectivity in each type of the models. The first interesting result is that the length of the randomly chosen path tended to present relatively small to moderate dispersion, enhancing the statistical relevance of the analysis performed for the ratios. Because of its intrinsic more sequential nature (starts as a cycle, a fact reflected in the obtained very high values of L), the WS was the network model more intensely affected by the path-star transformation. Contrariwise, very little changes were observed for the BA model, which involves a significant number of hubs concentrating the edges into stars and reducing the average shortest path length. Interestingly, the ER model responded in surprising ways to the path-star transformation. While the diameter and average shortest path lengths underwent relatively small changes after the path-star transformation, a substantial increase of average clustering coefficient was observed for all con-

figurations. This effect is explained by the tendency of the path-star transformation to complete triangles involving edges between non-subsequent nodes along the transformed path. The correlations between the ratios of change and the length of the transformed path were also analyzed, confirming the markedly distinct response of the WS networks to the transformation. The effect of the path-star transformation on the average clustering coefficient was also found to become more pronounced for smaller values of m (i.e. average degree). By quantifying the responses of the three categories of networks to the path-star transformation, the reported investigation allows this type of network modification to be applied in order to change (hopefully improve) specific features of given real-world networks.

Several are the possible future developments motivated by this work. To begin with, motivated by the relationship established between long paths and hubs (dual), it would be interesting to invest additional efforts in characterizing the longest path typically found in different

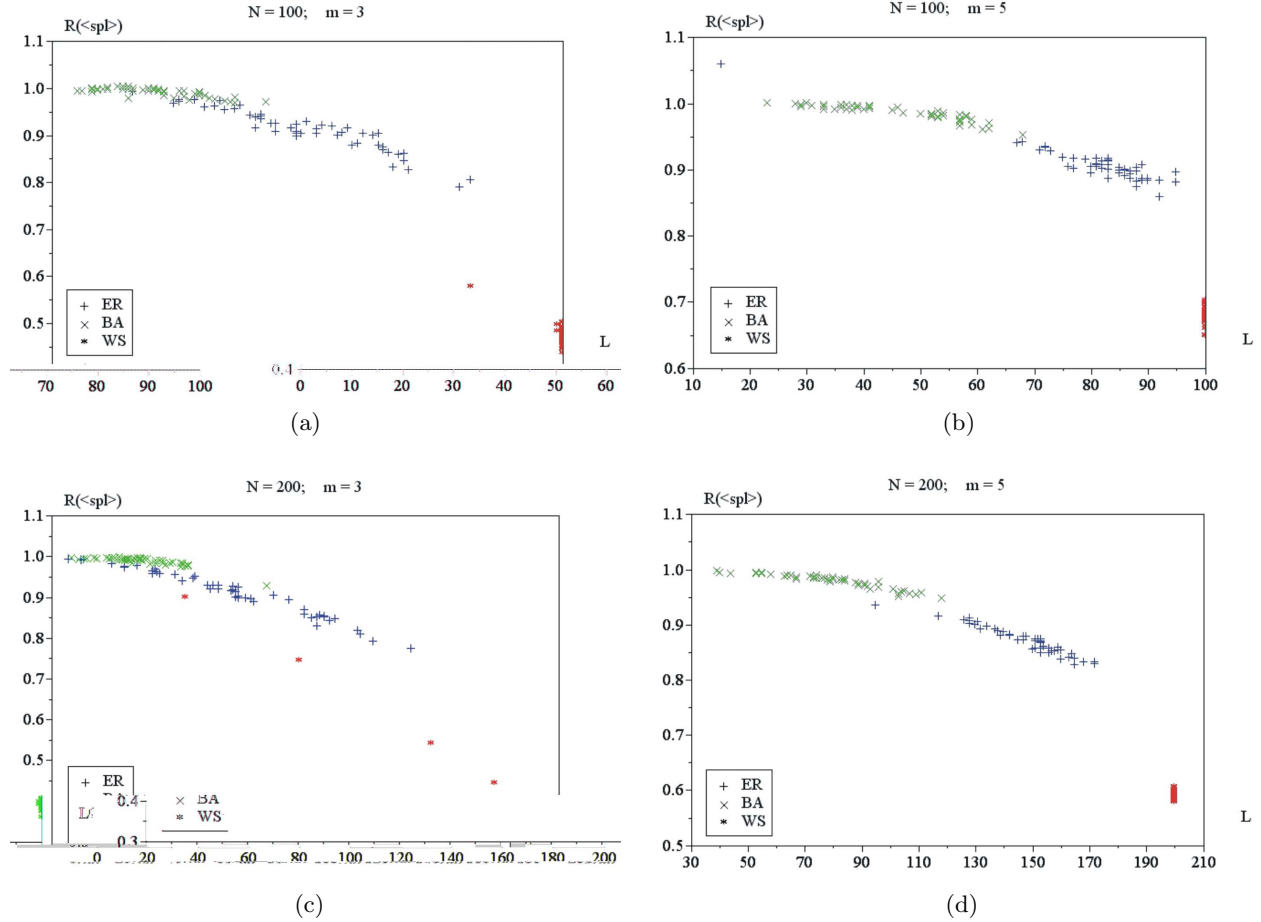


FIG. 4: The scatterplots obtained by plotting the ratio of change in the average shortest path length, $R(spl)$, in terms of L for the four considered configurations, i.e.: $N = 100, m = 3$ (a); $N = 100, m = 5$ (b); $N = 200, m = 3$ (c) and $N = 200, m = 5$ (d). These results indicate a clear tendency of $R(spl)$ to decrease with L in the cases of ER and BA networks. However, very little variation of $R(spl)$ was observed in the case of the WS model.

types of networks. Such a study would have immediate implications for the susceptibility of the models to the path-star transformation. Second, it would be natural to consider more than one subsequent path-star transformations. Other interesting future works include the consideration of real-world networks, the inverse star-path transformation, and the application of additional measurements (especially concentric features [7, 13]).

Acknowledgments

Luciano da F. Costa thanks CNPq (308231/03-1) and FAPESP (05/00587-5) for sponsorship.

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 - [15] A *path* in a network is a succession of adjacent edges in which all the edges and all the vertices are never repeated. The length of the path corresponds to its number of edges. (e.g. [14])
 - [16] Although paths are not typically considered motifs, as in [12] here we extend the concept of motifs to incorporate paths.